COHOMOLOGY OF ASSOCIATIVE ALGEBRAS AND SPECTRAL SEQUENCES

BY

KUNG-SING SHIH

M.S., University of Kyoto, Japan, 1940

AN ABSTRACT OF A THESIS

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF DOCTOR OF PHILOSOPHY IN MATHEMATICS
IN THE GRADUATE COLLEGE OF THE
UNIVERSITY OF ILLINOIS, 1953

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The purpose of this paper is to investigate the cohomology of associative algebras by means of spectral sequences. Our main applications lie in the case of tensor products of algebras which are important in the classification problem of algebras according to the vanishing of the cohomology groups.

Let A, B be algebras with identity elements 1_A, 1_B over a field F, and $P = A \underset{F}{\otimes} B$ their tensor product over F. We are going to study the relations between the cohomology groups of P and those of A and B. A and B can be naturally regarded as subalgebras of the algebra P by the identifications $a \otimes 1_B = a$ and $1_A \otimes b = b$, and $1 = 1_A \otimes 1_B$ is the identity element of P. Let M be a unitary two-sided P-module, and consider the graduated group $C = \sum_{n=0}^{\infty} C^n$, where $C^n = C^n(P, M)$ is the group of normalized n-cochains for P in M. The coboundary operator d is defined, as usual, by the formula:

$$df(p_1, \cdots, p_{n+1}) = p_1 \cdot f(p_2, \cdots, p_{n+1}) + \sum_{r=1}^{n} (-1)^r f(p_1, \cdots, p_r p_{r+1}, \cdots, p_{n+1}) + (-1)^{n+1} f(p_1, \cdots, p_n) \cdot p_{n+1}.$$

Define a filtration (C_j) of C as follows: $C_j = C$ for $j \leqq 0$. For $j > 0$, we set $C_j = \sum_{n=0}^{\infty} (C_j \cap C^n)$, where $C_j \cap C^n = (0)$ if $j > n$, and where, for $j \leqq n$, $C_j \cap C^n$ is the group of all n-cochains which vanish whenever $n - j + 1$ among the n arguments belong to the subalgebra A. Finally, we define B-operators on the group $C^n(A, M)$ by

$$(b.f)(a_1, \ldots, a_n) = b.f(a_1, \ldots, a_n), \qquad (f.b)(a_1, \ldots, a_n) = f(a_1, \ldots, a_n).b .$$

If we denote by (E_r) the spectral sequence derived from the filtration (C_j), our main theorem can be expressed by

$$E_1^{j,i} \approx C^j(B, H^i(A,M)), \qquad E_2^{j,i} \approx H^j(B, H^i(A,M)),$$

where the B-operators on $H^i(A, M)$ are induced in a natural way. This result is analogous to those which have been obtained for fibre spaces (Serre, [4]) and for group extensions (Hochschild - Serre, [2]).

In Part I, §§ 2 — § 4, we give a straight-forward proof of the main theorem. We introduce the following abbreviations: for $r \leqq s$, p_r^s and $a\left[\begin{smallmatrix} s \\ r \end{smallmatrix}\right]$ stand for the $(s - r + 1)$-tuple $(p_r, p_{r+1}, \cdots, p_s)$ of elements in P and the

continued product $a_r a_{r+1} \cdots a_s$ of $s - r + 1$ factors in A, respectively. We define, for $f \in C^{i+j-1}$ and an $(i + j)$-tuple (p_1^i, q_1^j) of elements in P, the partial coboundary operators δ_i and ∂_j by the formulas:

$$\delta_i f(p_1^i, q_1^j) = p_1 . f(p_2^i, q_1^j) + \sum_{r=1}^{i-1} (-1)^r f(p_1^r, p_r p_{r+1}, p_{r+2}^i, q_1^j) + (-1)^i f(p_1^{i-1}, q_1^j) . p_i$$

and

$$\partial_j f(p_1^i, q_1^j) = q_1 . f(p_1^i, q_2^j) + \sum_{r=1}^{j-1} (-1)^r f(p_1^i, q_1^r, q_r q_{r+1}, q_{r+2}^j) + (-1)^j f(p_1^i, q_1^{j-1}) . q_j .$$

Define, for $f \in C^{k+i+j}$ and a $(k + i + j)$-tuple (p_1^k, a_1^i, b_1^j) with $p_t \in P$, $a_r \in A$, $b_s \in B$, the value

$$f(p_1^k ; a_1^i, b_1^j) = (-1)^{ij} f(p_1^k ; b_1^j, a_1^i)$$

to be equal to the sum of $(i + j)$ terms $\pm f(p_1^k, r_1^{i+j})$, where the first k elements p_1^k are kept fixed, while the last $i + j$ arguments r_1^{i+j} run through all the permutations S of (a_1^i, b_1^j) which preserve the orders of the a's and b's, and where the sign is to be $+$ or $-$ according as the permutation S is even or odd. For the special case where $k = 0$, the value $f(; a_1^i, b_1^j)$ will also be denoted by $f_j(a_1^i, b_1^j)$. Then we have the following identity:

for $f \in C^{i+j-1}$, $a_r \in A$, $b_s \in B$,

$$(1) \qquad (df)_j(a_1^i, b_1^j) = \delta_i(f_j)(a_1^i, b_1^j) + (-1)^i \partial_j(f_{j-1})(a_1^i, b_1^j) .$$

Now consider the map: $f \in C_j \cap C^{i+j} \longrightarrow f_j' \in C^j(B, C^i(A,M))$ defined by

$$[f_j'(b_1^j)](a_1^i) = f_j(a_1^i, b_1^j) .$$

$f \longrightarrow f_j'$ induces a homomorphism $\sigma_j : C_j/C_{j+1} \longrightarrow C^j(B, C(A,M))$. Using the identity (1) we can deduce that $\sigma_j \circ d = d \circ \sigma_j$, and consequently, σ_j induces a homomorphism $\bar{\sigma}_j : E_1^j \ (\approx H(C_j/C_{j+1})) \longrightarrow C^j(B, H(A,M))$.

We have to show that $\bar{\sigma}_j$ is an isomorphism onto. The onto proof turns out to be a special case of the isomorphism proof. The latter forms the essential part of the entire theorem.

Choose an F-basis $(b^{(k)})$ of B such that $b^{(1)} = 1$. (Then $(b^{(k)})$ forms a free A-basis of P.) For arguments of the form $p_r = a_r b_r$, where $a_r \in A$, $b_r \in (b^{(k)})$, $1 \leqq r \leqq i + j - 1$, we define

(2) $$h(p_1^{i+j-1}) = (-1)^{i-1} \sum_{t=1}^{i} \left\{ (-1)^{(j-1)(t-1)}\, U^t + \sum_{s=0}^{j-1} (-1)^{s+t}\, V_s^t \right\}$$

where

$$U^1 = d\left[u(b_i^{i+j-1})\right](a_1^{i-1},\ a\begin{bmatrix} i+j-1 \\ i \end{bmatrix}) + (-1)^{i+1}\, u(b_i^{i+j-1})(a_1^{i-1}).a\begin{bmatrix} i+j-1 \\ i \end{bmatrix}$$

$$U^t = d\left[u(b_{i-t+1}^{i-t+j})\right](a_1^{i-t},\ a\begin{bmatrix} i-t+j \\ i-t+1 \end{bmatrix},\ a_{i-t+j+1}^{i+j-1}) \qquad 2 \leqq t \leqq i$$

$$V_0^t = f(\ a_1^{i-t},\ a\begin{bmatrix} i-t+j \\ i-t+1 \end{bmatrix};\ b_{i-t+1}^{i-t+j},\ a_{i-t+j+1}^{i+j-1}) \qquad 1 \leqq t \leqq i$$

$$V_s^t = f(p_1^{i-t+s},\ a\begin{bmatrix} i-t+j \\ i-t+s+1 \end{bmatrix};\ b_{i-t+s+1}^{i-t+j},\ a_{i-t+j+1}^{i+j-1}) \qquad \begin{matrix} 1 \leqq s \leqq j-1 \\ 1 \leqq t \leqq i \end{matrix}$$

and then extend, by linearity, the definition of h to arbitrary elements of P.

Now, under the assumptions that $f \in C_j \cap C^{i+j}$, $df \in C_{j+1}$, and $f(\ ;\ a_1^i,\ b_1^j) = d\left[u(b_1^j)\right](a_1^i)$, where $u \in C^j(B,\ C^{i-1}(A,M))$, we have to verify that $h \in C_j \cap C^{i+j-1}$ and $f - dh \in C_{j+1}$.

From the structure of the formula (2) and by simple combinatorial considerations, the verification of $h \in C_j \cap C^{i+j-1}$ is readily done. Regarding the proof of $f - dh \in C_{j+1}$, i. e. in order to show that, if i of the arguments p_1^{i+j} of $f - dh$ belong to A, then $(f - dh)(p_1^{i+j}) = 0$, we observe the followings: (i) if the last argument p_{i+j} does not belong to A, then, for each cochain h in the expansion of $(f - dh)(p_1^{i+j})$, all U^t and V_s^t with superscripts $t \neq 1$ vanish; (ii) if, for $0 < m < i$, just m last arguments belong to A, then all U^t and V_s^t with $t \neq m, m+1$ vanish; (iii) if the last i arguments belong to A, then all U^t and V_s^t with $t \neq i$ vanish. Furthermore, in each case, except for one extreme subcase, U^t and V_0^t always vanish (and in these exceptional cases the substitution of U^t yields $dd\left[u(\ )\right](\ ) = 0$), and eventually some of the first V_s^t vanish together. Thus, by actual substitution of the formula (2) in the expansion of $(f - dh)(p_1^{i+j})$ according to the coboundary formula, we shall have, in all cases, the following result: Let $(1 \leqq)\ r_1 < r_2 < \ldots < r_i\ (\leqq i+j)$ be the subscripts for the arguments belonging to A among p_1^{i+j}. (We have only to consider the case where the remaining arguments, call them q_s, do not belong to A.) Put $k = i - m$, where m $(0 \leqq m \leqq i)$ is the exact number of the elements of A occurring at the end of the $(i + j)$-tuple p_1^{i+j}. Then we have always:

(3) $$(f - dh)(p_1^{r_k}, q_{r_k+1}^{k+j}, a_{k+j+1}^{i+j})$$

$$= (-1)^{r_k-1} \sum_{s=0}^{k+j-r_k-1} (-1)^s (df)(p_1^{r_k}, q_{r_k+1}^{r_k+s}, a\begin{bmatrix} k+j \\ r_k+s+1 \end{bmatrix}; b_{r_k+s+1}^{k+j}, a_{k+j+1}^{i+j})$$

$= 0$, since $df \in C_{j+1}$. Here $p_t = \begin{cases} a_t & \text{for } t = r_1, \ldots, r_k \\ q_t & \text{for } t \neq r_1, \ldots, r_k \end{cases}$ $(1 \leqq t \leqq r_k)$

and $q_r = a_r b_r$ $(r_k + 1 \leqq r \leqq k + j)$.

From the isomorphism $\bar{\sigma}_j : E_1 \approx C(B, H(A,M))$, we shall proceed to derive $E_2 \approx H(B, H(A,M))$, using (1) once more and showing that the operator d_1 on E_1 is transformed into the coboundary operator on $C(B, H(A, M))$.

In I, § 5, we indicate how this direct proof may be carried over to the case of group extensions with appropriate modifications. The proof given in Chapter II of [2] is indirect in that it has recourse to another filtration.

In § 6, we give some applications of our result. For instance, we can deduce immediately, denoting by $K_m = K_m(F)$ the class of all algebras over a given field F whose m-dimensional cohomology groups are all zero, that from $A \in K_p$ and $B \in K_q$ follows $A \underset{F}{\otimes} B \in K_{p+q-1}$. This theorem was conjectured by Hochschild [1] and has been proved by Rose [3].

In Part II, which is independent of Part I, we start from a more general situation, namely, we assume that P is an algebra with an identity element 1 over a field F, and that P has a free (right) A-basis (b_k) such that $b_1 = 1$, where A is a subalgebra of P containing 1. This situation occurs, for example, when A is a simple subalgebra of a finite dimensional algebra P over F, as is well known in the theory of simple rings. We introduce (§ 1) a second filtration (C_j^*) as follows: $C_j^* = C$ for $j \leqq 0$. For $j > 0$, we set $C_j^* = \sum_{n=0}^{\infty} (C_j^* \wedge C^n)$, where $C_j^* \wedge C^n = 0$ if $j > n$, and where, for $j \leqq n$, $C_j^* \wedge C^n$ is the group of all elements $f \in C^n$ such that, for all $a \in A$, the followings hold:

$$f(p_1^{r-1}, p_r a, p_{r+1}^n) = f(p_1^r, a p_{r+1}, p_{r+2}^n)$$

for all r such that $n - j + 1 \leqq r \leqq n - 1$

and $$f(p_1^{n-1}, p_n a) = f(p_1^n).a .$$

This new filtration (C_j^*) enables us to proceed in quite a similar way as in

Chapter II of [2]. Thus we first show(§ 2) that the spectral sequence (E_r^*) derived from the filtration (C_j^*) coincides with (E_r) derived from the first filtration (C_j). By virtue of this lemma we may henceforth concentrate to the study of the groups E_r^*.

In § 3, we shall get the result:

$$E_1^{*j,i} \approx H^i(A,\ C_j^* \wedge C^j)\ ,$$

the isomorphism being induced by the homomorphism of C_j^*/C_{j+1}^* onto $C(A,\ C_j^* \wedge C^j)$, induced by the map: $f \in C_j^* \wedge C^{i+j} \longrightarrow {}_jf \in C^i(A,\ C_j^* \wedge C^j)$, where ${}_jf$ is defined by

$$[{}_jf(a_1^i)](p_1^j) = f(a_1^i\ ,\ p_1^j)$$

and A-operators on the group $C_j^* \wedge C^j$ are defined by

$$(a.g)(p_1^j) = a.g(p_1^j) \qquad \text{and} \qquad (g.a)(p_1^j) = g(\ ap_1,\ p_2^j) \qquad\qquad g \in C_j^* \wedge C^j.$$

As an application of this result, we shall prove, for the special case where the subalgebra A is separable, that the cohomology groups of P are equal to the relative cohomology groups of P modulo A (§ 4). This is a generalization of a theorem of Hochschild ([1], Theorem 10. 1), in which A was assumed to be a finite separable extension field over F .

Finally (§ 5), returning to the case of tensor products, we define a map $\phi : u \longrightarrow u'$ of $C^i(A,\ C_j^* \wedge C^j)$ into $C^j(B,\ C^i(A,M))$ by

$$[u'(b_1^j)](a_1^i) = [u(a_1^i)](b_1^j)\ .$$

Then it will be seen that ϕ is an isomorphism onto. Moreover ϕ commutes with the coboundary operators, so that it induces an isomorphism:

$$\bar{\phi} :\quad H^i(A,\ C_j^* \wedge C^j) \approx C^j(B,\ H^i(A,\ M))\ .$$

Combining this with the general result of § 3, we obtain our main theorem for E_1^* and then for E_2^* (and so, for E_1 and E_2).

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